

Computational Aspects of Asynchronous CA

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Abstract. This work studies some aspects of the computational power of fully asynchronous cellular automata (ACA). We deal with some notions of simulation between ACA and Turing Machines. In particular, we characterize the updating sequences specifying which are “universal”, i.e., allowing a (specific family of) ACA to simulate any TM on any input. We also consider the computational cost of such simulations.

Keywords: Asynchronous Cellular Automata, Computational complexity, Turing Machines.

1 Introduction

Cellular Automata (CA) are a computational model widely used in many scientific fields. A CA consists of identical finite automata arranged over a regular lattice. Each automaton updates its state on the basis of its own state and the one of its neighbors. All updates are synchronous.

CA are particularly successful for modelling real systems [2]. However, many natural systems have a clear asynchronous behavior (think of biological processes for example [19]). Asynchronous CA (ACA) have been introduced in order to be able to more closely simulate these systems. Roughly speaking, they are CA in which the constraint of synchronicity has been relaxed.

According to which updating policy is chosen, the behavior of the ACA under consideration can be very different. In literature, several policies have been considered (purely asynchronous [15], α -asynchronous [18], *etc.*)

In this paper we consider a fully asynchronous behavior in which, as in a continuous time process, two cells are never updated simultaneously (or, equivalently, one and only one cell is updated at each time step) [6,19]. Of course, the evolutions of such ACAs depend on the sequence of cells that are updated at each time step.

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It is well-known that CA are capable of universal computation (see [17] for an up-to-date survey). Various mechanisms have been introduced to show how ACA can emulate classical CA or circuits [15,16,11,21]. As a consequence of these results, Turing universality of ACA is also proved.

In this paper we focus on the way by which an ACA simulates a TM. We analyse two different modes: strict simulation and scattered strict simulation. Roughly speaking, strict simulations pretend that the ACA exactly reproduces the steps of the TM (up to some encoding) admitting at most that some time is wasted between two steps of the TM. The second mode is essentially the same as strict simulation but considers the case that only a subset of cells can be used to perform the simulation, the others being “inactive”. We characterize all updating schemes allowing these two simulation modes. Moreover, we show that in both cases, the time slowdown due to asynchronicity is quadratic w.r.t. the running time of the TM under simulation.

2 Basic Notions

Let A be an alphabet. A *CA configuration* (or simply configuration) is a function from \mathbb{Z} to A . A *cellular automaton* is a structure (A, r, λ) where A is the *alphabet*, r is the *radius*, and $\lambda : A^{2r+1} \rightarrow A$ is the *local rule*. The local rule is applied synchronously to all positions of a configuration. In other words, the local rule induces a global rule $f : A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$ defined as follows:

$$\forall c \in A^{\mathbb{Z}}, \forall i \in \mathbb{Z}, \quad f(c)_i = \lambda(c_{i-r}, \dots, c_{i+r})$$

Where $A^{\mathbb{Z}}$ is the set of all configurations and for each $i \in \mathbb{Z}$, the image $c(i)$ is denoted by c_i . Let λ be a local rule of radius r . Consider now the following asynchronous updating of a configuration. At each time t , f is applied on one and only one position. We denote by θ the sequence $(\theta_t)_{t \geq 0}$ whose generic element θ_t is the position which is updated at time t .

Definition 1. A fully asynchronous cellular automaton (ACA) is a quadruple (A, λ, r, θ) where A is a finite alphabet, $\lambda : A^{2r+1} \rightarrow A$ is the local rule of radius $r \in \mathbb{N}$ and $\theta = (\theta_t)_{t \geq 0}$, with $\theta_t \in \mathbb{Z}$ is a sequence of cell positions.

Every ACA $\mathcal{A} = (A, \lambda, r, \theta)$ induces a dynamical behavior described as follows. The evolution of any configuration $c \in A^{\mathbb{Z}}$ is the sequence of configurations

$$\{f^0(c), f^1(c), f^2(c), \dots, f^t(c), f^{t+1}(c), \dots\},$$

where $f^0(c) = c$ and, for any $t \in \mathbb{N}$, $f^{t+1}(c)$ is defined as

$$\forall i \in \mathbb{Z}, \quad f^{t+1}(c)_i = \begin{cases} \lambda(f^t(c)_{i-r}, \dots, f^t(c)_{i+r}) & \text{if } i = \theta_t \\ c_i & \text{otherwise} \end{cases}$$

We stress that dynamical evolutions in asynchronous CA depend on the choice of the updating policy. Different updating schemas have been considered for studying asynchronicity in CA settings [15,6,18]. We deal with the fully asynchronous

situation in which at each time only one cell is updated according to θ . Note that even for some relatively simple rules the behavior under fully asynchronous updating is far to be simple, see for instance [12,18]. From now on, for a sake of simplicity, by referring to ACA we will mean fully asynchronous CA.

A *Turing Machine* \mathcal{M} is a 7-tuple $(Q, \Sigma, \Gamma, \mathfrak{b}, \delta, q_0, F)$, where Q is the set of states, $\Sigma \subset \Gamma$ is the input alphabet, Γ is the working alphabet and $\mathfrak{b} \in \Gamma \setminus \Sigma$ is the blank symbol. The map $\delta : Q \times \Gamma \mapsto Q \times \Sigma \times \{L, R\}$ is the transition function, where L and R denote the left and right movements of the head, $q_0 \in Q$ is the initial state and $F \subseteq Q$ the set of final states (see [7], for an introduction on this subject). An *instantaneous configuration* c of \mathcal{M} is a triple (T, q, p) , where $T \in \Gamma^{\mathbb{Z}}$ is the content of the tape, $q \in Q$ is the current state of \mathcal{M} and $p \in \mathbb{Z}$ is the position of its head.

A *run* of \mathcal{M} on the initial input $x \in \Sigma^*$ is the sequence $R_t = \{(T_t, q_t, p_t)\}_{t \in \mathbb{Z}}$ where $(T_0, q_0, p_0) = (\omega \mathfrak{b} x \mathfrak{b}^\omega, q_0, 0)$ (i.e. the symbol \mathfrak{b} is repeated infinitely many times at the left and at the right of x) for $t = 0$, and for any $t \in \mathbb{N}$, $(T_{t+1}, q_{t+1}, p_{t+1})$ is the instantaneous configuration of \mathcal{M} at time $t + 1$, where T_{t+1} is equal to T_t except that in position p_t in which the symbol $(T_t)_{p_t}$ is replaced by the symbol s with $(q_{t+1}, s, X) = \delta(q_t, (T_t)_{p_t})$, where $p_{t+1} = p_t + 1$ if $X = R$ and $p_{t+1} = p_t - 1$ if $X = L$. Remark that if in a run it happens that at some time $t \in \mathbb{N}$ the state $q_t \in F$, then, for any $k > t$, $R_k = R_t$, i.e., in other words the computation halts on the instantaneous configuration R_t and the output is the non blank content of tape R_t .

3 Simulation of Turing Machines

It is well-known that CA are a universal computational model according to different notions of universality (for a survey see [17]). The main point to prove universality is to simulate a TM. Of course, one can apply similar ideas and constructions to prove computational universality or computational capability of ACA (see for example [15,16,11]). In this section we would like to precise the computational cost of such simulations.

The basic idea when simulating a TM using an ACA is to act by “extracting” first the information about the state of the TM from the current configuration of the ACA, and then to operate the TM transition saving information on the ACA configuration again. The way of saving the TM state and the way we extract it from the current configuration lead to the two following notions of simulation.

Notation. For $a, b \in \mathbb{N}$ with $a < b$, denote $[a, b]$ the set of integers between a and b (including a and b). Given the finite sets A_1, A_2, \dots, A_n , for any element $(a_1, a_2, \dots, a_n) \in A_1 \times A_2 \times \dots \times A_n$, define $\Pi_{A_i}((a_1, a_2, \dots, a_i, \dots, a_n)) = a_i$. These projection maps can be naturally extended to work with configurations, indeed given a configuration $c \in (A_1 \times \dots \times A_n)^{\mathbb{Z}}$, for any $i \in [1, n]$ and $j \in \mathbb{Z}$, define $\Pi_{A_i}(c)_j = \Pi_{A_i}(c_j)$.

Given a configuration $c \in A^{\mathbb{Z}}$ and a function $\psi : \mathbb{Z} \rightarrow \mathbb{Z}$, c^ψ is the configuration defined as $c_i^\psi = c_{\psi(i)}$ for all $i \in \mathbb{Z}$.

Definition 2. Let $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0, F)$ be a TM and $\mathcal{A} = (A, \lambda, r, \theta)$ be an ACA. \mathcal{A} strictly simulates \mathcal{M} iff $A = \Gamma \times B$ for some finite set B and for any input $x \in \Sigma^*$ of \mathcal{M} , there exists a configuration $c \in A^{\mathbb{Z}}$ satisfying the following conditions:

1. $\Pi_{\Gamma}(c) = \omega \mathfrak{b} x \mathfrak{b}^{\omega}$ and $\Pi_B(c) = \omega s u s^{\omega}$ for some $u, s \in B$;
2. for any time $t \in \mathbb{N}$, there exists $t' \in \mathbb{N}$ such that

$$\Pi_{\Gamma}(f^{t'}(c)) = T_t,$$

3. for any pair of times $t_1, t_2 \in \mathbb{N}$,

$$t_1 < t_2 \Rightarrow t'_1 < t'_2,$$

where $t'_i = \min\{k \in \mathbb{N} : \Pi_{\Gamma}(f^k(c)) = T_{t_i}\}$, $i = 1, 2$.

In other words, an ACA \mathcal{A} strictly simulates a TM \mathcal{M} if its configurations can represent in a direct way the tape of \mathcal{M} , possibly using an additional amount of information (stored in the alphabet B) and some additional time. Relaxing the condition on the representation of the tape, the following weaker notion of simulation is obtained.

Definition 3. Let $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0, F)$ be a TM. Let $\mathcal{A} = (A, \lambda, r, \theta)$ be an ACA. \mathcal{A} scattered strictly simulates \mathcal{M} iff $A = \Gamma \times B$ for some finite set B , and for any input $x \in \Sigma^*$ of \mathcal{M} there exist an injective increasing function $\psi : \mathbb{Z} \mapsto \mathbb{Z}$ and a configuration $c \in A^{\mathbb{Z}}$ satisfying the following conditions:

1. (a) $\Pi_B(c^{\psi}) = \omega s u s^{\omega}$, for some $u, s \in B$;
(b) $\forall i \in \mathbb{Z} \setminus \psi(\mathbb{Z}), \Pi_B(c_i) = q$, for some $q \in B$;
(c) $\Pi_{\Gamma}(c^{\psi}) = \omega \mathfrak{b} x \mathfrak{b}^{\omega}$;
2. for any time $t \in \mathbb{N}$, there exists $t' \in \mathbb{N}$ such that

$$\Pi_{\Gamma}((f^{t'}(c))^{\psi}) = T_t$$

3. for any pair of times $t_1, t_2 \in \mathbb{N}$,

$$t_1 < t_2 \Rightarrow t'_1 < t'_2,$$

where $t'_i = \min\{k \in \mathbb{N} : \Pi_{\Gamma}((f^k(c))^{\psi}) = T_{t_i}\}$, $i = 1, 2$.

A scattered strict simulation assumes that only a subset of cells participates to the simulation and the others are somehow inactive. For this reason, in the ACA configurations there can be an offset made for example by \mathfrak{b} s between the symbols of the TM tape content. Note that when the function ψ is $\psi(i) = i$ then scattered strict simulation and strict simulation coincide.

According to the above definitions, even if an ACA can simulate a TM on a fixed input x , it might not be able to simulate the same TM on a different input simply because of an inappropriate updating sequence θ .

3.1 Construction 1.

Given a TM $\mathcal{M} = (Q, \Sigma, \Gamma, \mathfrak{b}, \delta, q_0, F)$ build a family of ACA $\mathcal{A}_\theta = (A, \lambda, 1, \theta)$ such that $A = \Gamma \times Q \times D \times C$, where $D = \{L, R\}$, $C = \{0, 1, 2\}$, and the local rule $\lambda : A^3 \rightarrow A$ is defined as follows

$$\lambda(u, v, z) = \begin{cases} (\sigma, q, m, 0) & \text{if } u = (\sigma_u, q_u, R, 1), v = (\sigma_v, q_v, m_v, 2) \\ & \text{and } \delta(q_u, \sigma_v) = (q, \sigma, m) \\ (\sigma_v, q_v, R, 2) & \text{if } v = (\sigma_v, q_v, R, 1), z = (\sigma_z, q_z, m_z, 0) \\ (\sigma_v, q_v, m_v, 1) & \text{if } u = (\sigma_u, q_u, R, 2), v = (\sigma_v, q_v, m_v, 0) \\ (\sigma, q, m, 0) & \text{if } z = (\sigma_z, q_z, L, 1), v = (\sigma_v, q_v, m_v, 2) \\ & \text{and } \delta(q_z, \sigma_v) = (q, \sigma, m) \\ (\sigma_v, q_v, L, 2) & \text{if } v = (\sigma_v, q_v, L, 1), u = (\sigma_u, q_u, m_u, 0) \\ (\sigma_v, q_v, m_v, 1) & \text{if } z = (\sigma_z, q_z, L, 2), v = (\sigma_v, q_v, m_v, 0) \\ v & \text{otherwise.} \end{cases}$$

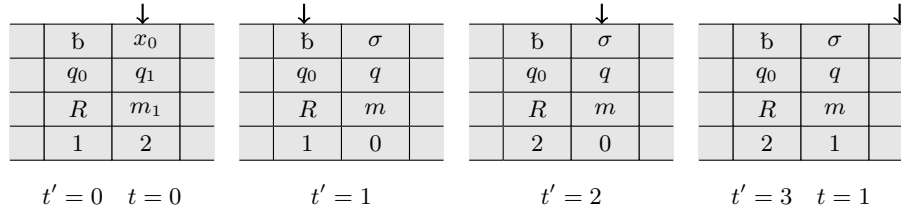


Fig. 1. Simulation of the first step of a TM using an ACA built by construction 1 with updating sequence $\theta = (0, -1, 0, 1, \dots)$. The ACA and TM times are denoted by t' and t , respectively. The arrow points at the current active cell of the ACA.

Every cell of the ACA contains the symbol of the corresponding cell on the TM tape, the state of the TM, the direction of movement of the TM head, and a value $\xi \in C$ to control the simulation. At TM time t , the cell i with $\xi = 1$ is the one where the TM head is positioned at time $t - 1$, i.e., $p_{t-1} = i$ (with $p_{-1} = -1$). During the ACA evolution, at most one cell in whole configuration has $\xi = 1$. If the updating sequence allows the cell $i + 1$ (resp., $i - 1$) to be updated and the cell i has $m = R$ (resp., $m = L$), then the cell $i + 1$ (resp., $i - 1$) changes its state according to the TM rule and its own value of ξ is set to 0 to indicate that the information about the head position has to be moved to this cell. To perform it, at subsequent times ACA will set the cell with $\xi = 1$ to 2 and the cell with $\xi = 0$ to 1. An example of this behavior is shown in Figure 1.

In order to (strictly) simulate a TM on input $x = x_0 \dots x_{n-1} \in \Sigma^*$, ACA given by the above construction have to start on the following configuration

$c \in A^{\mathbb{Z}}$:

$$\forall i \in \mathbb{Z}, c_i = \begin{cases} (x_i, q, m, 2) & \text{if } 0 \leq i < |x|. \\ (\mathfrak{b}, q_0, R, 1) & \text{if } i = -1. \\ (\mathfrak{b}, q, m, 2) & \text{otherwise,} \end{cases}$$

where q and m are an arbitrarily chosen state and movement (since they will not be used in the simulation, their choice can be arbitrary). The last point to precise is which updating sequences can be used. Of course, that depends on the TM to simulate but there are sequences that can be used in “all occasions”, they are called universal. An updating sequence θ is *universal* iff $|\{i \in \mathbb{Z}, \theta_i = k\}| = \infty$ for every $k \in \mathbb{Z}$; practically speaking a sequence is universal if any cell is updated infinitely many times.

Theorem 1. *An ACA $\mathcal{A} = (A, \lambda, 1, \theta)$ given by Construction 1 is Turing universal if and only if θ is universal.*

Proof. Consider a TM $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, \mathfrak{b}, q_0, F)$ and an ACA $\mathcal{A} = (A, f, 1, \theta)$ with θ universal. For any $x \in \Sigma^*$ input of \mathcal{M} , let c be the initial configuration built in construction 1. Let us prove that \mathcal{A} strictly simulates \mathcal{M} .

Let $R_t = (T_t, q_t, p_t)$ be the configuration of the \mathcal{M} at time t . We claim that for all $t \in \mathbb{N}$ there exists $t' \in \mathbb{N}$ such that the configuration $c' = f^{t'}(c)$ of \mathcal{A} has the following properties

1. $\Pi_{\Gamma}(c') = T_t$.
2. $\Pi_Q(c'_{p_{t-1}}) = q_t$ ($p_{-1} = -1$)
3. $\Pi_C(c'_{p_t}) = 1$ and $\Pi_C(c'_i) \neq 1$, for all $i \in \mathbb{Z} \setminus \{p_t\}$.
4. $\Pi_D(c'_{p_t}) = R$ if $p_t > p_{t-1}$; L if $p_t < p_{t-1}$.

We proceed by induction. For $t = 0$ the claim is true by construction ($t' = 0$ and $c' = c$). Assume that the claim is true for $t > 0$, *i.e.*, there exists t' such that the configuration $c' = f^{t'}(c)$ satisfies the four stated properties. Remark that $\Pi_C(c'_{p_t}) = 1$ and hence the only cells that can change their value are at positions $p_t + 1$ or $p_t - 1$, depending on the value of $\Pi_D(c'_{p_t})$. Assume that $\Pi_D(c'_{p_t}) = R$ (the other case is similar). Since θ is universal, there exists $t'' > t'$ such that $\theta_{t''} = p_t + 1$ and for any other $\bar{t} \in \mathbb{N}$ either $\theta_{\bar{t}} \neq p_t + 1$ or $\bar{t} > t''$. According to the definition of f , at time t'' the cell $p_t + 1$ will become $(\sigma, q, m, 0)$, where $(\sigma, q, m) = \delta(\Pi_Q(c''_{p_t}), \Pi_{\Gamma}(c''_{p_t+1}))$ and $c'' = f^{t''}(c)$. Moreover, no other cell can change its content between time $t' + 1$ and $t'' - 1$. Therefore $\Pi_{\Gamma}(c'') = T_{t+1}$ and $\Pi_Q(c''_{p_t+1}) = q_{t+1}$ *i.e.* c'' satisfies the first and second properties.

Again, since θ is universal, there exists $t''' > t''$ such that $\theta_{t'''} = p_t$ and for any other $\bar{t} \in \mathbb{N}$ either $\theta_{\bar{t}} \neq p_t$ or $\bar{t} > t'''$. According to f , the forth component of the cell at position $\theta_{t'''}$ is set to 2. Once more, remark that no changes in the configuration of the ACA occur between time $t'' + 1$ and $t''' - 1$.

Finally, by the universality of θ , there exists $\bar{t} > t'''$ such that $\theta_{\bar{t}} = p_t + 1$ and for any other $\bar{t} \in \mathbb{N}$ either $\theta_{\bar{t}} \neq p_t + 1$ or $\bar{t} > \bar{t}$. From the definition of f , one deduces that the only possibility is that the fourth component of the cell at

position $p_t + 1$ in $f^{\tilde{t}}(c)$ is set to 1. Since no changes in the configuration of the ACA occur between time $t''' + 1$ and $\tilde{t} - 1$, the claim is proved.

To prove the inverse implication, consider the TM

$$\mathcal{M} = (\{q_R, q_L\}, \{0, 1\}, \{0, 1, \mathfrak{b}\}, \mathfrak{b}, \delta, q_R, \emptyset)$$

where δ is defined as follows

$$\frac{(q, \sigma) \mid (q_R, \mathfrak{b}) \mid (q_R, 0) \mid (q_R, 1) \mid (q_L, \mathfrak{b}) \mid (q_L, 0) \mid (q_L, 1)}{\delta(q, \sigma) \mid (q_L, 1, L) \mid (q_R, 1, R) \mid (q_R, 1, R) \mid (q_R, 0, R) \mid (q_L, 0, L) \mid (q_L, 0, L)}$$

On any input, \mathcal{M} writes a symbol 1 on the cell 0, then it writes 1s towards the right until a blank symbol is reached. When a blank is reached, it moves left writing a symbol 0 until a blank is encountered at this point it starts moving right writing 1s and so forth. It is clear that the head of \mathcal{M} passes through any cell of the tape infinitely many times. Therefore any ACA given by Construction 1 needs an universal updating sequence to strictly simulate it. \square

The previous result proves that the class of ACA given by Construction 1 are computational universal but it seems that requiring an universal updating sequence involves a considerable time loss (see the proof of the Theorem1). The following proposition shows that there exist (carefully chosen) updating sequences such that the time loss is acceptable (quadratic).

Proposition 1. *Given a TM \mathcal{M} that executes in time $T(n)$, there exists an ACA \mathcal{A} given by Construction 1 that simulates \mathcal{M} in time $O(T(n)^2)$.*

Proof. Let s_i with $i \in \mathbb{N}$ be the finite sequence given by all the integers between $-i$ and $+i$ with step 2 (e.g., $s_2 = (-2, 0, 2)$ and $s_1 = (-1, 1)$). Consider the sequence θ given by the concatenation of the s_i sequences:

$$\theta = \underbrace{s_0(-1)s_0}_{s_1s_0s_1} \underbrace{s_1s_0s_1}_{s_2s_1s_2} \dots = \underbrace{0, -1, 0}_{s_1s_0s_1} \underbrace{-1, 1, 0, -1, 1}_{s_2s_1s_2} \dots$$

Clearly, θ is universal. Consider $\mathcal{A} = (A, \lambda, 1, \theta)$ where λ and A are as in Construction 1. It is easy to verify that every block $s_i s_{i-1} s_i$ simulates one step of \mathcal{M} . The size of the block increases by a (multiplicative) constant 3 for every i . The total length of the simulation is then bounded by

$$1 + |s_0| + \sum_{i=1}^{T(n)} 2 \cdot |s_i| + |s_{i-1}| = 2 + \sum_{i=1}^{T(n)} 3i + 2 = \frac{3}{2} \left(T(n)^2 + \frac{7}{3}T(n) + \frac{4}{3} \right) = O(T(n)^2).$$

Remark 1. The time $O(T(n)^2)$ is the better asymptotic limit for a sequence. Indeed, consider a sequence such that the corresponding ACA simulates in time $O(g(T(n))) \subseteq O(T(n)^2)$ any TM which executes in time $T(n)$. We show that $O(g(T(n))) = O(T(n)^2)$. Consider all the possible movement of the TM head. Its position is 0 at time 0, either 1 or -1 , at time 1, and so on. Let us introduce the graph $G = (V, E)$ where $V = \{0, \dots, T(n)\} \times \{-T(n), \dots, T(n)\}$ and $E = \{((t, a), (t+1, b)) \mid t \in \{0, \dots, T(n)-1\} \text{ and } |a-b| = 1\}$. Any path starting from

$(0,0)$ and ending to $(T(n), b)$ with $b \in \{-T(n), \dots, T(n)\}$ represents a possible sequence of a TM head. In order to simulate all the head movements, all the nodes of the graph must be visited at least once by these paths. Since the graph has $O(T(n)^2)$ vertexes, the time $O(g(T(n)))$ can not be less than $O(T(n)^2)$.

Remark 2. Actually, for any a TM \mathcal{M} which executes in time $O(T(n))$, there are uncountably many ACA given by Construction 1 that simulate \mathcal{M} in time $O(T(n)^2)$. An infinite set of them is individuated by the sequences obtained from the one illustrated in the Proposition 1 “inserting” in it any integer in one or more positions.

The slowdown in the simulation of TM using ACA given by Construction 1 is essentially given by the fact that we want a strict simulation and we must keep track (among other things) of the position of the head of the TM. Relaxing this last constraint brings to a different notion of simulation and to a different construction.

3.2 Construction 2

Given a TM $\mathcal{M} = (Q, \Sigma, \Gamma, \mathfrak{b}, \delta, q_0, F)$ build a family of ACA $\mathcal{A}_\theta = (A, \lambda, 1, \theta)$ such that $A = \Gamma \times Q \times D \times C$, where $D = \{L, R\}$, $C = \{1, 2\}$ and $\lambda : A^3 \rightarrow A$ is defined as follows

$$\lambda(u, v, z) = \begin{cases} (\sigma, q, m, 1) & \text{if } u = (\sigma_u, q_u, R, 1), \Pi_\Gamma(v) = \sigma_v, \\ & \Pi_C(z) \neq 1 \text{ and } \delta(q_u, \sigma_v) = (q, \sigma, m) \\ (\sigma', q', m', 1) & \text{if } z = (\sigma_z, q_z, L, 1), \Pi_\Gamma(v) = \sigma_v, \\ & \Pi_C(u) \neq 1 \text{ and } \delta(q_z, \sigma_v) = (q', \sigma', m') \\ (\Pi_\Gamma(v), \Pi_Q(v), \Pi_D(v), 2) & \text{otherwise.} \end{cases}$$

The initial configuration for starting the simulation is the same as the one given for Construction 1. As already remarked before, this family does not strictly simulate \mathcal{M} since it does not keep track of the head position. However, using similar techniques as in Theorem 1, one can prove that for $\theta = s_0 s_1 s_2 \dots$, the ACA $(A, \lambda, 1, \theta)$ simulates \mathcal{M} on any input with a total running time

$$\sum_{i=0}^{T(n)} |s_i| = \sum_{i=0}^{T(n)} (i+1) = \frac{1}{2} (T(n)^2 + 3T(n) + 2) = O(T(n)^2)$$

where $T(n)$ is the running time of \mathcal{M} on the input x .

3.3 Construction 3

Construction 1 and 2 assume that potentially all cells of the ACA cooperate to the simulation of the TM. Assume now that only a subset of cells participates to the simulation and the others are somehow inactive. In this section, we are going

to show that the ACA can still (scattered strictly) simulate any TM whenever the updating sequence has some specific properties.

A set $S \subset \mathbb{Z}$ is *syndetic* if there exists some finite $E \subset \mathbb{Z}$ such that $\cup_{n \in E} (S - n) = \mathbb{Z}$, where $(S - n) = \{k \in \mathbb{Z} \mid k + n \in S\}$. Syndetic sets have bounded gaps *i.e.* there exists $g \in \mathbb{N}$ (which depends on S) such that for any $h \in \mathbb{Z}$, $\{h, h+1, \dots, h+g\} \cap S \neq \emptyset$. Given a sequence $\alpha = \{\alpha_i\}_{i \in \mathbb{Z}}$, the *support* of α is the set $\text{supp}(\alpha) = \cup_{i \in \mathbb{Z}} \{i\}$.

Notation. To shorten up the notation in what follows, given an ordered sequence of states $u^{(-r)}, \dots, u^{(0)}, \dots, u^{(r)}$, denote $E_R(k) = \{i \in [1, r] \mid \Pi_C(u^{(i)}) = k\}$ and similarly $E_L(k) = \{i \in [-r, -1] \mid \Pi_C(u^{(i)}) = k\}$. Finally, denote $j_R = \min E_R(k)$ if $E_R(k) \neq \emptyset$ and $j_L = \max E_L(k)$ if $E_L(k) \neq \emptyset$.

Given a TM $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0, F)$ build a family of ACA $\mathcal{A}_\theta = (A, \lambda, r, \theta)$ such that $A = \Gamma \times Q \times D \times C$, $D = \{L, R\}$, $C = \{0, 1, 2, 3\}$. The local rule $\lambda : A^{2r+1} \rightarrow A$ is defined as follows

$$\lambda(u^{(-r)}, \dots, u^{(0)}, \dots, u^{(r)}) = \left\{ \begin{array}{ll} & \begin{array}{l} \text{if } E_L(1) \neq \emptyset, \\ (\sigma, q, m, 0) \quad \begin{array}{l} u^{(j_L)} = (\sigma_u, q_u, R, 1), \\ u^{(0)} = (\sigma_v, q_v, m_v, 2), \\ \text{and } \delta(q_u, \sigma_v) = (q, \sigma, m) \end{array} \end{array} \\ & \begin{array}{l} \text{if } u^{(0)} = (\sigma_v, q_v, R, 1), \\ (\sigma_v, q_v, R, 2) \quad \begin{array}{l} E_R(0) \neq \emptyset, \\ \text{and } u^{(j_R)} = (\sigma_z, q_z, m_z, 0) \end{array} \end{array} \\ & \begin{array}{l} \text{if } E_L(2) \neq \emptyset, \\ (\sigma_v, q_v, m_v, 1) \quad \begin{array}{l} u^{(j_L)} = (\sigma_u, q_u, R, 2), \\ u^{(0)} = (\sigma_v, q_v, m_v, 0) \end{array} \end{array} \\ & \begin{array}{l} \text{if } E_R(1) \neq \emptyset, \\ (\sigma, q, m, 0) \quad \begin{array}{l} u^{(j_R)} = (\sigma_z, q_z, L, 1), \\ u^{(0)} = (\sigma_v, q_v, m_v, 2) \\ \text{and } \delta(q_z, \sigma_v) = (q, \sigma, m) \end{array} \end{array} \\ & \begin{array}{l} \text{if } u^{(0)} = (\sigma_v, q_v, L, 1), \\ (\sigma_v, q_v, L, 2) \quad \begin{array}{l} E_L(0) \neq \emptyset, \\ u^{(j_L)} = (\sigma_u, q_u, m_u, 0) \end{array} \end{array} \\ & \begin{array}{l} \text{if } E_R(2) \neq \emptyset, \\ (\sigma_v, q_v, m_v, 1) \quad \begin{array}{l} u^{(j_R)} = (\sigma_z, q_z, L, 2), \\ u^{(0)} = (\sigma_v, q_v, m_v, 0) \end{array} \end{array} \\ & u^{(0)} \quad \text{otherwise.} \end{array} \right.$$

In order to be able to (scattered strictly) simulate a TM on input $x_0 \dots x_{n-1} \in \Sigma^*$, ACA given by the above construction have to be started on the following

configuration c

$$\forall i \in \mathbb{Z}, c_i^\alpha = \begin{cases} (\mathfrak{b}, q_0, R, 1) & \text{if } i = \alpha_{-1} \\ (x_i, q, m, 2) & \text{if } i \in [\alpha_0, \alpha_{n-1}] \\ (\mathfrak{b}, q, m, 2) & \text{if } i \in \text{supp}(\alpha) \setminus [\alpha_0, \alpha_{n-1}] \\ (\mathfrak{b}, q, m, 3) & \text{otherwise} \end{cases}$$

where α is a subsequence of θ such that $\alpha_0 < \alpha_1 < \dots < \alpha_n$, and $q \in Q$, $m \in D$ are arbitrarily chosen. Clearly, the whole construction (and hence the simulation) depends on θ and its subsequence α . The following result characterizes them.

Theorem 2. *An ACA $\mathcal{A} = (A, \lambda, r, \theta)$ given by Construction 3 scattered strictly simulates any TM on any input if and only if θ contains an universal subsequence α such that $\text{supp}(\alpha)$ is a syndetic set.*

Proof. For any TM $\mathcal{M} = (Q, \Sigma, \Gamma, \mathfrak{b}, \delta, q_0, F)$ consider an ACA $\mathcal{A} = (A, \lambda, r, \theta)$ given by Construction 3. Assume that \mathcal{A} scattered strictly simulates \mathcal{M} . First of all, let us prove that $|\text{supp}(\theta)| = \infty$. Indeed, if $|\text{supp}(\theta)| < \infty$, only a finite number of cells can be used for simulation and therefore, according to Condition 2 of Definition 3 only a finite portion of the tape can be simulated. Choose a subsequence α of θ such that $\alpha_0 < \alpha_1 < \dots < \alpha_n$ (this is possible since $|\text{supp}(\theta)| = \infty$). By contradiction, assume that no α is universal. This means that the set of cells that can be updated infinitely many times is finite or empty. Without loss of generality assume that it has finite cardinality and $j > 0$ is the maximal of its elements (the case $j \leq 0$ is similar). Let k be the index of last occurrence of j in α . Consider the TM \mathcal{M} from the proof of Theorem 1 on the empty input. Since, for all $t > k$, $f^t(c^\alpha)_j = c_j^\alpha$, Condition 2 of Definition 3 is violated.

Now, always by contradiction, assume that there exist universal subsequences of θ but none of them has a syndetic support set. This means that there are larger and larger sets $[a, b] \subset \mathbb{N}$ not contained in $\text{supp}(\alpha)$. Choose one of them such that $b - a > r$ and let \mathcal{M} be the TM which on the empty input writes $2b$ symbols 1 as an output. Set $h = \min_{j \in \text{supp}(\alpha)} \{b < j\}$. According to the definition of λ , for all $t \in \mathbb{N}$, $f^t(c^\alpha)_h = c_h^\alpha$. Hence Condition 2 of Definition 3 is false. Therefore if \mathcal{A} scattered strictly simulates \mathcal{M} , θ has to contain an universal subsequence whose support is syndetic.

On the other hand, assume that θ contains an universal subsequence α with syndetic support. Then, there exists a subsequence α' with $\alpha'_0 < \alpha'_1 < \dots < \alpha'_n$, where n is the length of the input of \mathcal{M} . Build the initial configuration $c^{\alpha'}$ as described in Construction 3. Since $\text{supp}(\alpha)$ has bounded gaps, let p be the shortest one and set the radius $r = p$. The rest of the proof is essentially the same as the one given for Theorem 1 with α' playing the role of θ . \square

Proposition 2. *For any TM \mathcal{M} that executes in time $T(n)$, consider an ACA \mathcal{A} given by Construction 3 and an updating sequence θ containing an universal subsequence whose support is syndetic. Then, \mathcal{A} scattered strictly simulates \mathcal{M} in time $O(T(n)^2)$.*

Proof. Consider $\mathcal{A} = (A, \lambda, p, \theta)$ where λ and A are as in Construction 3. For $i \in \mathbb{N}$, let s_i be as in the proof of Proposition 1. Let α be the subsequence of θ given in the hypothesis and let $p \in \mathbb{N}$ be the minimal gap. Consider the subsequence $\alpha' = s_{2p}s_{4p} \dots s_{2ip} \dots$. Similarly to the proof of Proposition 1, $s_{2ip}s_{4ip}s_{6ip}$ can be used to encode the simulation of a step of \mathcal{M} . The total length of the simulation is then bounded by

$$\sum_{i=1}^{T(n)} |s_{2ip}| + |s_{4ip}| + |s_{6ip}| = \sum_{i=1}^{T(n)} 12ip + 3 = 6p \left(T(n)^2 + \left(1 + \frac{1}{2p}\right) T(n) \right) = O(T(n)^2).$$

□

4 Future work

This paper studies the simulation of Turing Machines by fully asynchronous CA. We have shown that computational universality for ACA corresponds to the condition that the ACA updating sequence contains any cell infinitely many times (universal sequence). We have also exhibited some universal sequences in order that the computational cost of the simulation is reasonable (quadratic) w.r.t the length of the run of the simulated TM.

The construction of sequences that allow quadratic simulations might seem artificial, classifying our results in the domain of computability but of no use in practical simulations. Indeed, for example, consider the program described in [1]. It is used to simulate biochemical processes in cells. It is essentially based on an ACA (although the authors do not clearly state this) which represents proteins (and other chemicals) by particles. The current configuration is updated in two steps: diffusion and collision/reaction. In the diffusion step a particle is randomly chosen, a direction is randomly chosen and then the particle makes a move in this direction. Adopting a more “system” based vision instead of a particle based one and passing in 1D for simplicity sake, one can represent the sequence of activations of the different sites (which may contain a particle or not) as $\theta_0 = 0, \theta_1 = \theta_0 + X_1, \theta_2 = \theta_1 + X_2, \dots, \theta_t = \theta_{t-1} + X_t, \dots$, where X_i are random variables with values in $\{-1, +1\}$ identically distributed (with uniform Bernoulli distribution for example). Therefore, the current active cell $\theta_t = \sum_{i=0}^t X_i$ is also a random variable with values in \mathbb{Z} and, practically speaking, the updating sequence is a random walk (see [10] for more on random walks). It is well-known that 1D random walks pass through all sites infinitely many times, therefore θ is an universal updating sequence and hence, by Theorem 1, the system described in [1] is capable of Turing universal computation.

For every alphabet A , local rule λ and radius r it is possible to consider the class of all the ACA such in the form (A, λ, r, θ) where θ is generated by a 1D random walk. Then it is possible to investigate for any property the probability of finding an ACA inside that class that has the given property. In the case of simulation of Turing machines it can be proved that for any TM \mathcal{M} running in time $T(n)$, there exists an ACA with updating sequences generated by random

walks such that the probability of (strictly) simulating \mathcal{M} in time $3T(n)$ is $2^{-3T(n)}$. The above result says that there exist a class of ACA where a particular ACA \mathcal{A} chosen uniformly from that class has a low probability of simulating a TM faster than deterministic ACA seen in the paper. The authors are currently investigating to see if and up to which extent the above results can be improved.

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References

1. Patrick Amar, Gilles Bernot, and Victor Norris. Hsim: a simulation programme to study large assemblies of proteins. *Journal of Biological Physics and Chemistry*, 4:79–84, 2004.
2. B. Chopard. Modelling physical systems by cellular automata. In G. Rozenberg et al., editor, *Handbook of Natural Computing: Theory, Experiments, and Applications*. Springer, 2011. To appear.
3. M. Cook. Universality in elementary cellular automata. *Complex Systems*, 15:1–40, 2004.
4. M. Delorme and J. Mazoyer. Cellular automata as languages recognizers. In Marianne Delorme and Jacques Mazoyer, editors, *Cellular Automata, Mathematics and Its Applications*. Kluwer, Dordrecht, 1999.
5. J.-C. Dubacq. How to simulate turing machines by invertible one dimensional cellular automata. *IJFOCS*, 6:395–402, 1995.
6. N. Fatès, M. Morvan, N. Schabanel, and E. Thierry. Fully asynchronous behaviour of double-quiescent elementary cellular automata. *Theoretical Computer Science*, 362:1–16, 2006.
7. J. E. Hopcroft and J. D. Ullman. *Introduction to Automata Theory, Languages and Computation*. Addison-Wesley, 1979.
8. A. Ray Smith III. Simple computation-universal cellular spaces. *Journal of the ACM*, 18:339–353, 1971.
9. J. Kari. Theory of cellular automata: A survey. *Theoretical Computer Science*, 334:3–33, 2005.
10. J. F. C. Kingman. *Poisson Processes*. Oxford University Press, 1993.
11. J. Lee, S. Adachi, F. Peper, and S. Mashiko. Delay-insensitive computation in asynchronous cellular automata. *J. Comput. Syst. Sci.*, 70:201–220, 2005.
12. J. Lee, S. Adachi, F. Peper, and K. Morita. Asynchronous game of life. *Physica D*, 194:369–384, 2004.
13. K. Lingren and M. G. Nordahl. Universal computation in simple one-dimensional cellular automata. *Complex Systems*, 4:299–318, 1990.
14. M. Mitchell. Computation in cellular automata: A selected review. In *Nonstandard Computation*, pages 95–140. Wiley-VCH, 1996.
15. K. Nakamura. Asynchronous cellular automata and their computational ability. *Systems, Computers, Control*, 5:58–66, 1974.
16. C. L. Nehaniv. Evolution in asynchronous cellular automata. *Artificial Life VIII*, pages 65–73, 2002.

17. N. Ollinger. Universalities in cellular automata. In G. Rozenberg et al., editor, *Handbook of Natural Computing: Theory, Experiments, and Applications*. Springer, 2011. To appear.
18. D. Regnault, N. Schabanel, and E. Thierry. Progresses in the analysis of stochastic 2d cellular automata: A study of asynchronous 2d minority. *Theoretical Computer Science*, 410:4844–4855, 2009.
19. B. Schönfisch and A. de Roos. Synchronous and asynchronous updating in cellular automata. *BioSystems*, 51:123–143, 1999.
20. K. Sutner. A note on culik-yu classes. *Complex Systems*, 3:107–115, 1989.
21. T. Worsch. A note on (intrinsically?) universal asynchronous cellular automata. Preprint, 2010.